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The difference between optimality and universality

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Abstract

We discuss the difference between optimality and universality. The sequence of measures of a universal test is well studied. To analyze the sequence of measures of an optimal Martin-Löf test, we introduce uniform Solovay reducibility. Solovay reducibility is a measure of relative randomness between two reals. In contrast uniform Solovay reducibility is a measure of relative randomness between two sequences of reals. Finally we prove that a sequence is uniform Solovay complete iff it is the sequence of measures of an optimal Martin-Löf test.

keywords: Martin-Löf randomness, randomness deficiency, sequence of measures, universal Martin-Löf test, Solovay reducibility

1 Introduction

Martin-Löf randomness [11] is regarded as the most natural randomness notion. A universal Martin-Löf test can be a collection of open sets $\{U_n : n \in \omega\}$ with $U_n \supset U_{n+1}$ and $\mu(U_n) \leq 2^{-n}$. A sequence X is random if it passes the test, meaning that $X \notin \bigcap_n U_n$. Thus a non-random sequence A will leave the test at some point. It is natural to associate a measure of non-randomness to be the least n where $A \notin U_n$. This was called the *critical level* by Martin-Löf and the randomness deficiency by Levin [8]. From a statistical point of view we are interested in the size of the set of sequences whose randomness deficiency are larger than a constant. Note that it is the sequence of measures of the test.

There is a notion of *universal* ML-test meaning that if X passes the test then it will be random, and a notion of *optimal* test, meaning that the randomness deficiency of the test is within a constant of any other test. For example, the *standard* universal is constructed by $U_n = \bigcup_n U^e_{n+e+1}$ where $\{U^e_n : n \in \omega\}$ is an effective enumeration of all ML-tests. This will be universal. It is not clear what the difference (if any) is between universality and optimality, and the main idea of the present paper is to explore the difference between these two notions.

The sequence of measures of a universal ML-test is studied by Kučera and Slaman [7] and Merkle, Mihailović and Slaman [12], which use Solovay reducibility. To give a version of an optimal ML-test, we introduce uniform Solovay reducibility. Solovay reducibility is a measure of relative randomness between two reals. In contrast uniform Solovay reducibility is a measure of relative randomness between two sequences of reals. We shall show that there is a uniform Solovay complete sequence. Finally we prove that the sequence of measures of an optimal ML-test can be characterized by a uniform Solovay complete sequence, which is the main result of this paper.

In Section 2 we give some definitions and results related with ML-randomness. In Section 3 we prove some results on randomness deficiency and an optimal ML-test. In Section 4 we characterize the sequence of measures of an optimal ML-test by introducing uniform Solovay reducibility.

2 Preliminaries

2.1 Notations

We fix notations used in this paper and recall some basic definitions and results. For a more complete introduction, see Downey and Hirschfeldt [4] or Nies [13].

A set of natural numbers is *computable* if its characteristic function is computable. A set of natural numbers is *c.e.* if it is the domain of a partial computable function. A sequence of sets $A_n \subseteq \mathbb{N}$ is uniformly c.e. if $\{\langle n, m\rangle : m \in A_n\}$ is c.e.

Let $2^{<\omega}$ be the set of all finite binary strings. A natural number n is identified with a string σ such that 1σ is the binary representation of $n+1$. Let 2^ω be Cantor space of binary sequences of $\{0,1\}$. We identify a set of natural numbers with a binary sequence. We use λ to denote the empty string. For $\sigma \in 2^{<\omega}$, $|\sigma|$ denotes the length of σ. We write $\sigma\rho$ to mean the concatenation of σ and ρ. We write $\sigma \prec \tau$ to mean that σ is a prefix of τ, that is $(\exists\rho)\sigma\rho = \tau$. Here τ can be infinite. Let $[\![\sigma]\!] = \{Z \in 2^\omega : \sigma \prec Z\}$ be the class of infinite binary sequences extending σ. We assume that 2^ω is equipped with the topology generated by the base $\{[\![\sigma]\!] : \sigma \in 2^{<\omega}\}$. For $A \subseteq 2^{<\omega}$ we let $[\![A]\!] = \bigcup_{\sigma\in A}[\![\sigma]\!]$. An open set A is *c.e.* if the corresponding set of strings $\{\sigma : [\![\sigma]\!] \subseteq A\}$ is a c.e. set.

We also identify real numbers with their inifinite binary expansion. Then elements of Cantor space 2^ω are sometimes called *reals*. We say that a real α is *left-c.e.* or *c.e.* if $\{q \in \mathbb{Q} : q < \alpha\}$ is c.e. A function $f : 2^{<\omega} \to \mathbb{R}$ is *c.e.* if the sequence $\{\langle q, \sigma\rangle : q < f(\sigma)\}$ is c.e. We denote the *uniform or Lebesgue measure* by μ, that is generated by $\mu([\![\sigma]\!]) = 2^{-|\sigma|}$.

2.2 ML-randomness

First we recall the definition of Martin-Löf randomness and related results.

Definition 2.1 (Martin-Löf [11])**.** *A* Martin-Löf test *(or* ML-test*) is a sequence of uniformly c.e. open sets $\{U_n\}$ such that $\mu(U_n) \le 2^{-n}$. A real α* passes *a ML-test $\{U_n\}$ if $\alpha \notin \bigcap_n U_n$. A real α is* ML-random *or* 1-random *if α passes all ML-tests.*

Martin-Löf randomness is also characterized by complexity. A *machine* is a partial computable function. There is a *universal* machine, i.e., a machine V

such that for each M there is a string $\tau \in 2^{<\omega}$ for which $(\forall\sigma)V(\tau\sigma) = M(\sigma)$ or both $V(\tau\sigma)$ and $M(\tau)$ diverge. The *plain Kolmogorov complexity* C of a string is defined as $C(\sigma) = \min\{\tau \ : \ V(\tau) = \sigma\}$ where V is a universal machine. A set X is *prefix-free* if whenever $\sigma, \tau \in X$, then σ is not a proper prefix of τ. A machine $M : 2^{<\omega} \to 2^{<\omega}$ is called a *prefix-free machine* if $\mathrm{dom}(M)$ is prefix-free. There is a universal prefix-free machine U. Then *prefix-free Kolmogorov complexity* K of a string σ is defined as $K(\sigma) = \min\{\tau \ : \ U(\tau) = \sigma\}$.

Proposition 2.2. *A real α is ML-random iff $(\exists d)(\forall n)K(\alpha\restriction n) > n - d$.*

Another characterization is given by martingales. A function $d : 2^{<\omega} \to \mathbb{R}^{\geq 0}$ is a *martingale* if for all σ, $2d(\sigma) = d(\sigma 0) + d(\sigma 1)$. It is a *supermartingale* if for all σ, $2d(\sigma) \geq d(\sigma 0) + d(\sigma 1)$. A (super)martingale d *succeeds* on a sequence α if $\limsup_n d(\alpha \restriction n) = \infty$.

Proposition 2.3 (Schnorr [14])**.** *A real α is ML-random iff no c.e. martingale succeeds on alpha iff no c.e. supermartingale succeeds on α.*

By these equivalences Martin-Löf randomness is regarded as a natural notion of randomness.

2.3 Universality and Optimality

In most recent papers in algorithmic randomness, we usually use the following definition for universality.

Definition 2.4. *A ML-test $\{U_n\}$ is universal if, for any Martin-Löf test $\{V_n\}$, $\bigcap_n V_n \subseteq \bigcap_n U_n$.*

In the original Martin-Löf's paper, however, he defined a "universal" ML-test in a different way. To distinguish them we change the terminology.

Definition 2.5 (Martin-Löf [11])**.** *A ML-test $\{U_n\}$ is optimal if, for any ML-test $\{V_n\}$, there exists c such that $V_{n+c} \subseteq U_n$ for all n.*

The terminology "optimal" comes from the tradition for martingales. Optimality is a stronger notion than universality. As is well-known, such a test exists.

Proposition 2.6 (Martin-Löf [11])**.** *There exists an optimal Martin-Löf test. Hence there exists an universal Martin-Löf test.*

2.4 Randomness deficiency

Martin-Löf also introduced the critical level, the smallest level of significance on which the hypothesis is rejected. Levin [8] called it *randomness deficiency.*

Definition 2.7 (Martin-Löf [11])**.** *Let $U = \{U_n\}$ be a ML-test. We define*

$$t_U(\sigma) = \sup\{n \ : \ [\![\sigma]\!] \subseteq U_n\}$$

for $\sigma \in 2^{<\omega}$ *and*

$$t_U(\alpha) = \sup\{n \ : \ \alpha \in U_n\}$$

for $\alpha \in 2^{\omega}$.

Intuitively randomness deficiency indicates how much regularity it contains. For simplicity let $U_0 = [\![\lambda]\!] = 2^{\omega}$. Then the following are immediate.

Proposition 2.8 (Martin-Löf [11])**.** (i) $0 \le t_U(\sigma) \le |\sigma|$.

(ii) $t_U(\sigma) \le t_U(\tau)$ *for all* $\sigma \preceq \tau$.

(iii) $t_U(A) = \sup_n t_U(A \restriction n)$.

(iv) *If* A *is ML-random, then* $t_U(A) < \infty$.

(v) *If* $\{U_n\}$ *is an optimal ML-test then, for any ML-test* $V = \{V_n\}$*, there exists* c *such that* $t_U(\sigma) \ge t_V(\sigma) - c$ *for all* $\sigma \in 2^{<\omega}$.

As usual we let $t = t_U$ for an optimal ML-test U.

Ten years after [8], Levin [9] called another quantity randomness deficiency. In most articles the terminology of randomness deficiency is used for this quantity.

Definition 2.9 (Levin [9]; see [10])**.** *Randomness deficiency* δ *is defined as* $\delta(\sigma) = |\sigma| - K(\sigma)$.

2.5 From a statistical point of view

Randomness deficiency is closely related to probability theory. The strong law of large numbers (SLLN) in probability theory says that the average of a sequence of i.i.d. converges to the expectation [6, 5]. For a coin-tossing game, the relative frequency of a random binary sequence converges to $1/2$ almost surely. For an effective version of SLLN, the relative frequency of a Martin-Löf random sequence "always" converges to $1/2$, not almost surely [16, 10]. For a more precise version, Davie [2] proved the following result. Let $S_n(A) = \sum_{i=1}^{n} A(i)$. Then there exists a computable function $n(c,\epsilon)$ such that if $\sup_n \delta(A \restriction n) \le c$ then for all $n > n(c,\epsilon)$, we have $|S_n(A)/n - 1/2| < \epsilon$. This result says that we can discuss the rate of the convergence of SLLN from randomness deficiency δ. Davie [2] also showed a version of the law of the iterated logarithm.

We reconsider an optimal ML-test from a statistical point of view. Let H be the hypothesis that $A \in 2^{\omega}$ is a result of a fair coin-tossing game. Suppose that all we know is $S_{20}(A) = 15$. Then can we accept H or should we reject H? Since the probability of $S_{20} \ge 15$ is $0.0207\cdots$ and is small, probably we should reject it.

Next suppose that we know A itself. Then the probability of A is clearly 0 for each A. So we can not use the same approach. Recall that an optimal ML-test $\{U_n\}$ is one of the best effective statistical tests. Intuitively the larger $t(A)$ is, the more unnatural A is. Then we can reject H if $t(A)$ is too large.

Then the following question arises. Can we regard the ordering of naturalness as probability? We wish the probability of $t(A) \geq n$ were equal to 2^{-n}. We say that a ML-test $V = \{V_n\}$ is decreasing if $V_n \supseteq V_{n+1}$ for all n. Note that the measure of $t_V(A) \geq n$ for a decreasing optimal ML-test $V = \{V_n\}$ is

$$\mu(\{A : t_V(A) \geq n\}) = \mu(\{A : A \in V_n\}) = \mu(V_n).$$

We are very interested in the sequence $\mu(V_n)$ for an optimal ML-test. Then the question is where there is an optimal ML-test such that $\mu(U_n) = 2^{-n}$.

Such a test is called Schnorr test [14]. Formally a Schnorr test is a ML-test $\{U_n\}$ with $\mu(U_n) = 2^{-n}$. A Schnorr test $\{U_n\}$ is universal if, for any Schnorr test $\{V_n\}$, $\bigcap_n V_n \subseteq \bigcap_n U_n$. It is known that no Schnorr test is universal. Hence there is no universal ML-test such that $\mu(U_n)$ is computable uniformly in n. So $\mu(U_n) \neq 2^{-n}$ for infinitely many n.

Furthermore the following characterization is known. The sequence $\mu(U_n)$ for a universal ML-test $\{U_n\}$ is characterized by the following two theorems.

Theorem 2.10 (Kučera and Slaman [7])**.** *The measure of each component of a universal ML-test is ML-random.*

Theorem 2.11 (Merkle, Mihailović and Slaman [12])**.** *For any uniformly c.e. ML-random reals $r_n \leq 2^{-n}$ there is a universal ML-test $\{U_n\}$ such that $\mu(U_n) = r_n$.*

Since optimality implies universality, $\mu(V_n)$ is ML-random for an optimal ML-test V and each n.

Then we ask whether there is a universal or optimal ML-test $\{U_n\}$ such that $\mu(U_n) = 2^{-n}\alpha$ for a c.e. ML-random real α. Finally we show that for each c.e. ML-random real there exists a universal ML-test satisfying the equation but no optimal ML-test satisfies the equation. To prove this we introduce uniform Solovay reducibility.

3 Randomness deficiency by an optimal test

In this section we give basic results related to t in Definition 2.7 and an optimal ML-test. We will use some of them later.

3.1 Difference between universality and optimality

We prove the difference between optimality and universality. The following results say that the measures $\mu(U_n)$ of an optimal ML-test is approximately equal to 2^{-n} but the measures $\mu(U_n)$ of a universal ML-test may be far from it.

Proposition 3.1. *Let $\{U_n\}$ be an optimal ML-test. Then $\mu(U_n) \geq 2^{-n-O(1)}$.*

Proof. Let $V_n = [0^n]$. Then $V = \{V_n\}$ is a ML-test. Note that $t_V(0^n) = n$. By optimality of U there exists c such that $t_U(0^n) \geq t_V(0^n) - c = n - c$. By the definition of t we have $[0^n] \subseteq U_{n-c}$ for $n \geq c$. It follows that $2^{-n} = \mu([\![0^n]\!]) \leq \mu(U_{n-c})$. $\square$

Proposition 3.2. *There exists a ML-test such that it is universal but not optimal.*

Proof. Let $\{U_n\}$ be a universal Martin-Löf test. Then $V = \{V_n\} = \{U_{2n}\}$ is also a universal Martin-Löf test and we have $\mu(V_n) = \mu(U_{2n}) \leq 2^{-2n}$. By proposition 3.1, V is not optimal. $\square$

3.2 Decreasing optimal test

Another possible randomness deficiency is

$$\tilde{t}_U(\sigma) = \min\{n : [\![\sigma]\!] \subseteq U_n \text{ does not hold. }\} - 1.$$

The degree is infinity if such an n does not exist. We will prove t and $\tilde{t}$ are essentially the same. The following is a well-known result but we restate it in our terminology.

Proposition 3.3. *For a ML-test $U = \{U_n\}$, let $V_n = \bigcup_{k=1}^{\infty} U_{n+k}$ for $n \geq 1$ and $V_0 = [\![\lambda]\!]$. Then $V = \{V_n\}$ is a decreasing ML-test such that $t_U(\sigma) \leq t_V(\sigma) + 1$.*

Proof. We have $\mu(V_n) \leq \sum_{k=1}^{\infty} \mu(U_{n+k}) \leq \sum_{k=1}^{\infty} 2^{-n-k} \leq 2^{-n}$. Since $\{U_n\}$ is uniformly c.e., so is $\{V_n\}$. Hence $V = \{V_n\}$ is a ML-test. Note that for all n, we have $U_{n+1} \subseteq V_n$. For each $\sigma \in 2^{<\omega}$, $[\![\sigma]\!] \subseteq U_n \Rightarrow [\![\sigma]\!] \subseteq V_{n-1}$. Hence $t_U(\sigma) \leq t_V(\sigma) + 1$. Finally $V_n = \bigcup_{k=1}^{\infty} U_{n+k} \supseteq \bigcup_{k=2}^{\infty} U_{n+k} = \bigcup_{k=1}^{\infty} U_{n+1+k} = V_{n+1}$. Hence V is decreasing. $\square$

Proposition 3.4. *There exists a decreasing optimal Martin-Löf test.*

If V is decreasing, then $t_V = \tilde{t}_V$. Even if it is not, $W_n = \bigcup_{k=1}^{\infty} V_{n+k}$ is a decreasing ML-test. Hence a decreasing optimal ML-test U is also optimal for $\tilde{t}$. By letting $\tilde{t} = \tilde{t}_U$ we have $\tilde{t} = t + O(1)$. Thus these are essentially the same.

3.3 Relation with another randomness deficiency

We give easy relations between t and δ to use it the next section.

It is easy to see that there exists c such that

$$\delta(\sigma) \leq t(\sigma) + c$$

for all σ.

Let $\delta(\alpha) = \sup_n \delta(\alpha \upharpoonright n)$.

Theorem 3.5. *There exists c such that*

$$t(\alpha) \leq \delta(\alpha) + 2\log(\delta(\alpha)) + c$$

for all $\alpha \in 2^\omega$

Before the proof, recall KC Theorem.

Theorem 3.6 (KC Theorem, see [4]). *Let (d_i, τ_i) be a computable sequence of pairs (which we call requests), with $d_i \in \mathbb{N}$ and $\tau_i \in 2^{<\omega}$, such that $\sum_i 2^{-d_i} \leq 1$. Then there is a prefix-free machine M and strings σ_i of length d_i such that $M(\sigma_i) = \tau_i$ for all i and $\mathrm{dom} M = \{\sigma_i\}$*

The *weight* of a request (d, τ) is 2^{-d}. The *weight* of a computable sequence of requests (d_i, τ_i) is the sum of the weights of the requests, i.e., $\sum_i 2^{-d_i}$. If this weight is less than or equal to 1, then we say that this sequence is a *KC set*.

Proof of Theorem 3.5. Let $U = \{U_n\}$ be a decreasing optimal ML-test. Then there exists a uniformly c.e. prefix-free set R_n such that $[\![R_n]\!] = U_{n+2\log n}$. Let $L_c = \{\langle |x_i^n| - n + c, x_i^n\rangle \ : \ x_i^n \in R_n\}$. Then the weight of L_c is

$$\sum_n \sum_i 2^{-|x_i^n|+n-c} \leq \sum_n 2^{-n-2\log n+n-c} = 2^{-c} \sum_n n^{-2}.$$

Hence $L = L_c$ is a KC-set for some c. By KC Theorem we obtain $K(x_i^n) \leq |x_i^n| - n + c'$ and $\delta(x_i^n) \geq n - c'$ for some c'.

Suppose that $t(\alpha) \geq m + 2\log m$. Then $\alpha \in U_{m+2\log m} = [\![R_m]\!]$. It follows that there exists i such that $x_i^m \in R_m$ and $x_i^m \preceq \alpha$. Then $m \leq \delta(\alpha) + c'$. Hence we have $t(\alpha) \leq \delta(\alpha) + c' + 2\log(\delta(\alpha) + c') \leq \delta(\alpha) + 2\log(\delta(\alpha)) + c''$. □

Corollary 3.7. *Let $\{\alpha_k\}$ be a sequence of reals. Then*

$$\sup_k t(\alpha_k) < \infty \iff \sup_k \delta(\alpha_k) < \infty.$$

4 Uniform Solovay reducibility

In this section we generalize Solovay reducibility to analyze the sequence $\mu(U_n)$ where $\{U_n\}$ is an optimal ML-test. Solovay reducibility is a measure of relative randomness between two reals. We introduce uniform Solovay reducibility, which is a measure of relative randomness between two sequences of reals. Most proofs in the next subsection are just a generalization of the proof of corresponding results of Solovay reducibility. But in some points we need a little modification. So we concentrate on the difference and give a proof sketch in other points (see [4] for the detail).

A real α is *Solovay reducible* to a real β (written $\alpha \leq_s \beta$) if there are a constant c and a partial computable function $f : \mathbb{Q} \to \mathbb{Q}$ such that if $q \in \mathbb{Q}$ and $q < \beta$, then $f(q) \downarrow < \alpha$ and $\alpha - f(q) < c(\beta - q)$.

Let $\Omega_U = \sum_{U(\sigma)\downarrow} 2^{-|\sigma|}$. This is called halting probability. A left-c.e. real α is *Solovay complete or Ω-like* if $\beta \leq_s \alpha$ for all left-c.e. reals β.

Theorem 4.1 (Solovay [15], Calude, Hertling, Khoussainov, and Wang [1], Kučera and Slaman [7], see 9.1 and 9.2 in [4]). *For left-c.e. reals α, the following are equivalent.*

(i) *α is 1-random.*

(ii) *α is Solovay complete.*

(iii) *$\Omega \leq_s \alpha$.*

(iv) *$\alpha = \mu(U_n)$ for a universal ML-test $\{U_n\}$ and some n.*

Lemma 4.2 (Solovay [15])**.** *For each k there is a constant c_k such that for all n and all $\sigma, \tau \in 2^n$, if $|0.\sigma - 0.\tau| < 2^{k-n}$, then $C(\sigma) = C(\tau) \pm c_k$ and $K(\sigma) = K(\tau) \pm c_k$.*

4.1 Definition and some properties

In the following we consider a sequence of reals in $[0, 1]$.

Definition 4.3. *A sequence $\{\alpha_k\}$ of reals is uniformly Solovay reducible, or US-reducible, to a sequence $\{\beta_k\}$ of reals (written $\{\alpha_k\} \leq_{us} \{\beta_k\}$) if there are a constant c and uniformly partial computable functions $f_k : \mathbb{Q} \to \mathbb{Q}$ such that $f_k(0) = 0$ and if $q \in \mathbb{Q}$ and $q < \beta_k$, then $f_k(q) \downarrow < \alpha_k$ and $\alpha_k - f_k(q) < c(\beta_k - q)$.*

Note that the condition $f_k(0) = 0$ is equivalent to $\alpha_k < c\beta_k$ for some c. Hence $\{\alpha_k\} \leq_{us} \{\beta_k\}$ requires α_k/β_k is bounded. Solovay reducibility which considers only two reals does not need such a condition.

The definition of uniformly Solovay reducibility is not restricted to a sequence of uniformly left-c.e. reals. But here we consider only the uniformly left-c.e. reals.

Note also that us-reducibility is reflexive and transitive.

Proposition 4.4. *Let $\{\alpha_k\}$ and $\{\beta_k\}$ be sequences of uniformly left-c.e. reals, and let $0 = q_0^k < q_1^k < \cdots \to \alpha_k$ and $0 = r_0^k < r_1^k < \cdots \to \beta_k$ be uniformly computable sequences of rationals. Then $\{\alpha_k\} \leq_{us} \{\beta_k\}$ iff there are a constant c and uniformly computable functions g_k such that $\alpha_k - q^k_{g_k(n)} < c(\beta_k - r_n^k)$ for all k and n.*

Note that we require $q_0^k = r_0^k = 0$, which does not appear in a version of Solovay reducibility.

Proof. For the only if direction, given n, we have $f_k(r_n^k) \downarrow < \alpha_k$ and let $g_k(n) = m$ such that $f_k(q_n^k) < q_m^k$. For the if direction, given $s \in \mathbb{Q}$, search for an n such that $s < r_n^k$ and let $f_k(s) = q^k_{g_k(n)}$. □

Theorem 4.5. *If $\{\alpha_k\} \leq_{us} \{\beta_k\}$ then $(\exists d)(\forall k)(\forall n) C(\alpha_k \restriction n) \leq C(\beta_k \restriction n) + d$ and $(\exists d)(\forall k)(\forall n) K(\alpha_k \restriction n) \leq K(\beta_k \restriction n) + d$.*

Proof. Here we see $\beta_k \restriction n$ as rationals. Since $\beta_k - \beta_k \restriction n \leq 2^{-n}$, we have $\alpha_k - f_k(\beta_k \restriction n) < c2^{-n}$. Let $\tau_k^n \in 2^n$ be such that $|\tau_k^n - f(\beta_k \restriction n)| < 2^{-n}$. Then for all k,

$$|\alpha_k \restriction n - \tau_k^n| \leq |\alpha_k - \alpha_k \restriction n| + \alpha_k - f_k(\beta_k \restriction n) + |\tau_k^n - f_k(\beta_k \restriction n)| < (c+2)2^{-n}.$$

By Lemma 4.2, $K(\alpha_k \restriction n) \leq K(\tau_k^n) + O(1)$. Here τ_k^n can be obtained computably from $\beta_k \restriction n$, so $K(\tau_k^n) \leq K(\beta_k \restriction n) + O(1)$. The proof for plain complexity is the same. □

Lemma 4.6. *Let $\{\alpha_k\}$ and $\{\beta_k\}$ be sequences of uniformly left-c.e. reals, and let $0 = r_0^k < r_1^k < \cdots \to \beta_k$ be uniformly computable sequences of rationals. Then $\{\alpha_k\} \le_{us} \{\beta_k\}$ iff there are uniformly computable sequences of rationals $0 = p_0^k < p_1^k < \cdots \to \alpha_k$ such that for some constant c we have $p_s^k - p_{s-1}^k < c(r_s^k - r_{s-1}^k)$ for all k and s.*

Again note that we require $p_0^k = r_0^k = 0$ for all k.

Proof. If there is a sequence p_s^k as in the lemma, then $\alpha_k - p_n^k = \sum_{s>n}(p_s^k - p_{s-1}^k) < d\sum_{s>n}(r_s^k - r_{s-1}^k) = d(\beta_k - r_n^k)$, so by Proposition 4.4, $\{\alpha_k\} \le_{us} \{\beta_k\}$. We now prove the converse.

Let $0 = q_0^k < q_1^k < \cdots \to \alpha_k$ be uniformly computable sequences of rationals, let c and g_k be as in Proposition 4.4. We may assume without loss of generality that g is increasing. Note that $q_{g_k(0)}^k = r_0^k = 0$.

There must be an $s_0 > 0$ such that $q_{g_k(s_0)}^k - q_{g_k(0)}^k < c(r_{s_0}^k - r_0^k)$, since otherwise we would have $\alpha_k - q_{g_k(0)}^k = \lim_s(q_{g_k(s_0)}^k - q_{g_k(0)}^k) \ge \lim_s c(r_s^k - r_0^k) = c(r_k - r_0^k)$, contradicting our choice of c and g. We can now define $p_1^k, \ldots, p_{s_0}^k$ so that $p_0^k < \cdots < p_{s_0}^k = \alpha_{k,g_k(s_0)}$ and $p_s^k - p_{s-1}^k \le c(r_s^k - r_{s-1}^k)$ for all $s \le s_0$. See [4] for the detail.

We can repeat the procedure in the previous paragraph with s_0 in place of 0 to obtain a computable sequence of rationals $0 = p_0^k < p_1^k < \ldots$ with the desired properties. □

Theorem 4.7. *Let $\{\alpha_k\}$ and $\{\beta_k\}$ be sequences of uniformly left-c.e. reals, The following are equivalent.*

(i) $\{\alpha_k\} \le_{us} \{\beta_k\}$.

(ii) *For any uniformly computable sequences $b_1^k, b_2^k, \cdots$ of non-negative rationals such that $\beta_k = \sum_n b_n^k$, there are a constant c and uniformly computable sequences of rationals $\epsilon_{k,n} \in [0, c]$ for all n such that $\alpha_k = \sum_n \epsilon_{k,n} b_{k,n}$ for all k.*

(iii) *There are a constant c and uniformly left-c.e. reals γ_k such that $c\beta_k = \alpha_k + \gamma_k$ for all k.*

Note that the sequence b_n^k starts from $n = 1$. We may think that $b_0^k = 0$ for all k.

Proof. (i)⇒(ii) Let $b_1^k, b_2^k, \cdots$ be computable sequences of non-negative rationals such that $\beta_k = \sum_i b_i^k$ and let $r_n^k = \sum_{i \le n} b_i^k$. Note that $r_0^k = 0$. Apply Lemma 4.6 to obtain c and $p_0^k, p_1^k, \cdots$ as in that lemma. Let $\epsilon_n^k = \frac{p_n^k - p_{n-1}^k}{b_n^k}$. Then $\sum_n \epsilon_n^k b_n^k = \sum_n (p_n^k - p_{n-1}^k) = \alpha$, and $\epsilon_n = \frac{p_n - p_{n-1}}{r_n - r_{n-1}} \in [0, c]$ for all $n \ge 1$.

(ii)⇒(iii) Let $b_1^k, b_2^k, \cdots$ be computable sequences of non-negative rationals such that $\beta_k = \sum_n b_n^k$. Let ϵ_n^k be as in (ii), and let $\gamma_k = \sum_n (c - \epsilon_n^k) b_n^k$.

(iii)⇒(i) Let $0 = r_0^k < r_1^k < \cdots \to \alpha_k$ and $0 = s_0^k < s_1^k < \cdots \to \gamma_k$ be computable sequences. Let $p_n^k = \frac{r_n^k + s_n^k}{d}$. Then $0 = p_0^k < p_1^k < \cdots \to \beta_k$ and

$\alpha_k - r_n^k = \alpha_k + s_n^k - cp_n^k < \alpha_k + \gamma_k - cp_n^k = c(\beta_k - p_n^k)$, so by Proposition 4.4, $\{\alpha_k\} \le_{us} \{\beta_k\}$. □

4.2 Existence of a us-complete sequence

We say α_k is uniform Solovay complete (us-complete) if $\{\beta_k\} \le_{us} \{\alpha_k\}$ for all uniformly left-c.e. sequences $\{\beta_k\}$ of reals. We prove existence of a us-complete sequence.

Let M_i be an effective enumeration of all prefix-free machines.

Definition 4.8. *Let*

$$\Omega_k = \sum_{i=0}^{\infty} \sum_{M_i(0^k 1\sigma)\downarrow} 2^{-i-1-k-|\sigma|}.$$

Although the definition is very artificial, this sequence of Omegas has very natural properties as follows. The notation Ω_k comes from (ii) in the proposition below.

Proposition 4.9. (i) *The sequence $\{\Omega_k\}$ consists of uniformly left-c.e. reals.*

(ii) $\Omega_k \le 2^{-k}$.

(iii) $\sum_{k=0}^{\infty} \Omega_k = \Omega_U$ *for a universal prefix-free machine U.*

Proof. Note that $\Omega_k \le \sum_i 2^{-i-1-k} = 2^{-k}$. Furthermore

$$\sum_{k=0}^{\infty} \Omega_k = \sum_i \sum_{M_i(\sigma)\downarrow, \sigma \ne 0^n} 2^{-i-1-|\sigma|} = \Omega_U$$

where $U(0^i 1\sigma) = M_i(\sigma)$ for all σ such that $\sigma \ne 0^n$. Note that there is an i such that $M_i(0^n)\uparrow$ for all n and M_i is universal. Then U is also universal. □

Theorem 4.10. $\sup_k t(2^k \Omega_k) < \infty$.

Remark 4.11. *Since $\Omega_k \le 2^{-k}$, the first n-bits of $\Omega_k \in 2^\omega$ are 0. Then $2^k \Omega_k \in 2^\omega$ is the result of n-times left-shifts for Ω_k.*

Proof. For each k there is an i such that $V(\sigma) = M_i(0^k 1\sigma)$ is universal. Then $\Omega_V = \sum_{M_i(0^k 1\sigma)\downarrow} 2^{-k-|\sigma|}$ is 1-random and so is Ω_k. It follows that Ω_k is not dyadic rational. Hence we can assume that there is an s with $\Omega_{k,s} \restriction n = \Omega_k \restriction n$ for each k. Let U be a universal prefix-free machine defined in the proof of Proposition 4.9. We define $\Omega_{k,s} = \sum_{i=0}^{\infty} \sum_{U[s](0^i 1 0^k 1\sigma)\downarrow} 2^{-i-1-k-|\sigma|}$.

We build a prefix-free machine $M = M_c$. By the recursion theorem, we can assume we know its coding constant c in U. At a stage s the construction proceeds as follows. Search τ such that $U(\tau)[s] = 2^k \Omega_{k,s} \restriction n$ and $|\tau| < n - c$ for some k and n. Note that this means that $K(2^k \Omega_{k,s} \restriction n) < n - c$. If such τ is found, we choose a string $\mu \notin \mathrm{rng} U[s]$ and declare $M(0^k 1\tau) = \mu$. If such τ is not found, go to the next stage.

We see that this construction is valid. Let $\nu = 0^c10^k1\tau$. By the definition of U, we have $U(\nu) = M(0^k1\tau) = \mu$. Since $\mu \notin \mathrm{rng}U[s]$, it follows that $\nu \notin \mathrm{dom}U[s]$. Thus $\Omega_k - \Omega_{k,s} \geq 2^{-c-1-k-|\tau|} \geq 2^{-n-k}$ and $2^k\Omega_k - 2^k\Omega_{k,s} \geq 2^{-n}$. Hence $2^k\Omega_k \restriction n \neq 2^k\Omega_{k,s} \restriction n$. This procedure ensures that if $|\tau| < n - c$ then $U(\tau) \neq 2^k\Omega_k \restriction n$, whence $K(2^k\Omega_k \restriction n) \geq n - c$ for all k and n. By Corollary 3.7, $\sup_k t(2^k\Omega_k) < \infty$. □

Theorem 4.12. *The sequence $\{2^k\Omega_k\}$ is us-complete.*

Proof. Let α_k be uniformly left-c.e. reals. Then there exists an j such that $\alpha_k = \sum_{M_j(0^k1\sigma)\downarrow} 2^{-|\sigma|}$. Then

$$2^{j+1}2^k\Omega_k = \alpha_k + \sum_{i\neq j} \sum_{M_i(0^k1\sigma)\downarrow} 2^{-i-1-|\sigma|+j+1},$$

so $\{\alpha_k\} \leq_{us} 2^k\{\Omega_k\}$ by Theorem 4.7. □

4.3 Measures of an optimal ML-test

We will show here a version of an optimal ML-test of Theorem 2.11 and 2.10. Note that our proof is much simpler than those of versions of universality. This means that optimality is a more natural concept than universality.

Theorem 4.13. *Let r_n be uniformly left-c.e. reals such that $r_n \leq 2^{-n}$. Then the followings are equivalent.*

(i) *$\{2^nr_n\}$ is us-complete.*

(ii) *There exists an optimal ML-test U_n such that $\mu(U_n) = r_n$.*

Proof. (i)⇒(ii). Let $\{\Omega_n\}$ be the one defined above. Suppose that $\{2^n\Omega_n\} \leq_{us} \{2^nr_n\}$. Let $u_n = \mu(U_n)$ where $\{U_n\}$ is an optimal ML-test. Then there exists another optimal ML-test $\{V_n\}$ such that $\mu(V_n) = v_n = \sum_{m=1}^{\infty} m \cdot u_{n+2m}$ by adding extra strings. Note that $v_n \leq 2^{-n}\sum_{m=1}^{\infty} m \cdot 2^{-2m} \leq 2^{-n}$. By Theorem 4.12 we have $\{2^nv_n\} \leq_{us} \{2^n\Omega_n\} \leq_{us} \{2^nr_n\}$. Hence $\{2^nv_n\} \leq_{us} \{2^nr_n\}$ and $\{v_n\} \leq_{us} \{r_n\}$. Then by Theorem 4.7 we have $dr_n = v_n + \gamma_n$ for some d and uniformly left-c.e. reals γ_n.

It follows that $dr_n = \sum_{m=1}^{\infty} m \cdot u_{n+2m} + \gamma_n$ and $r_n = u_{n+2d} + \sum_{m\neq d} \frac{m}{d} \cdot u_{n+2m} + \gamma_n$. Hence we can construct a ML-test $\{W_n\}$ such that $W_n \supseteq U_{n+2d}$ and $\mu(W_n) = r_n$ by adding extra strings. Furthermore $\{W_n\}$ is optimal by $W_n \supseteq U_{n+2d}$.

(ii)⇒(i). Let $\{U_n\}$ be an optimal ML-test and $r_n = \mu(U_n)$. Let α_n be a uniformly left-c.e. sequence of reals with $\alpha_n \leq 2^{-n}$. We will show $\{\alpha_n\} \leq_{us} \{r_n\}$.

For each $m < n$, we shall construct a c.e. open set A_n^m in stages s. At stage s act as follows. If $A_n^m[s] \not\subseteq U_m[s]$ then do nothing. Otherwise let t be the last stage at which we put anything into A_n^m (or $t = 0$ if there is no such stage). Enumerate into A_n^m a set of strings $\{\sigma_i\}$ such that the set is prefix-incomparable (so that $[\![\sigma_i]\!]$ are pairwise disjoint), $\sum_i 2^{-|\sigma_i|} = 2^{2(m-n)}(\alpha_{m,s} - \alpha_{m,t})$ and $\sigma_i \notin \mathrm{dom} U_m[s] \cup A_n^m[s]$. We have $\mu(A_n^m) \leq 2^{2(m-n)}\alpha_m$.

Let $A_n = \bigcup_{m<n} A_n^m$ then $\mu(A_n) \leq \sum_{m<n} 2^{2(m-n)}\alpha_m \leq \sum_{m<n} 2^{m-2n} = 2^{-n}$. Hence $\{A_n\}$ is a ML-test. By optimality of U_n there exists d such that $A_{n+d} \subseteq U_n$ for some d independent on n. It follows that $U_n \supseteq A_{n+d} = \bigcup_{m<n} A_{n+d}^m \supseteq A_{n+d}^n$. Hence we enumerate something into A_{n+d}^n infinitely many times and $\mu(A_{n+d}^n) = \alpha_n$. Let $s_0 = 0$ and let $s_1, s_2, \cdots$ be the stages at which we put something into A_{n+d}^n. Then for $i > 0$ we have $r_{n,s_{i+1}} - r_{n,s_i} > 2^{-2d}(\alpha_{n,s_i} - \alpha_{n,s_{i-1}})$, so $\{\alpha_n\} \leq_{us} \{r_n\}$. ∎

4.4 The difference between optimality and universality

Theorem 4.13 gives us another difference between optimality and universality.

Theorem 4.14. *For a c.e. ML-random real $\alpha < 1$, there exists a universal ML-test $\{U_n\}$ such that $\mu(U_n) = 2^{-n}\alpha$. However no optimal ML-test $\{V_n\}$ satisfies $\mu(U_n) = 2^{-n}\alpha$ for a real α.*

The former statement is immediate from Theorem 2.11. The latter one follows from Theorem 4.13 and the following theorem.

Theorem 4.15. *The sequence $\{\alpha_k\}$ such that $\alpha_k = \alpha$ for all k is not us-complete.*

Proof. Suppose that $\{\alpha\}$ is us-complete. Let K be the halting set and $\beta_k = K(k) \in \{0,1\}$. Note that β_k are uniformly left-c.e. Since $\{\beta_k\} \leq_{us} \{\alpha\}$, we can approximate β_k within $1/2$ by using the first finite bits of α. It follows that the finite bits solve the halting problem, which is a contradiction. ∎

A similar difference can be seen for c.e. martingales. The collection of all sequences on which d succeeds is called the *success set* of d, and is denoted by $S[d]$. A c.e. (super)martingale d is *universal* if for any c.e. (super)martingale f, we have $S[f] \subseteq S[d]$. A c.e. (super)martingale d is *optimal* if, for each (super)martingale f, there is a constant c such that $c \cdot d(\sigma) \geq f(\sigma)$ for all σ.

Proposition 4.16 (Schnorr [14])**.** *There is a universal c.e. martingale.*

Proposition 4.17 (Downey, Griffiths, and LaForte [3])**.** *There is no optimal c.e. martingale.*

The author believes that there is some relation and we need further study.

Discussion

As is seen in Theorem 4.1, α is Solovay complete iff α is 1-random. Then does us-reducibility have a similar characterization? One may conjecture that $\{\alpha_k\}$ is us-complete iff $\{\alpha_k\}$ is 1-random. The natural definition of 1-randomness of a sequence would be $\bigoplus_k \alpha_k = \{\langle k,n\rangle \ : \ n \in \alpha_k\}$ is 1-random. Unfortunately this is false because a us-complete sequence satisfy $\alpha_k \geq \epsilon$ for some $\epsilon > 0$ by Theorem 4.13 and by Proposition 3.1 but, for such a sequence, $\bigoplus_k \alpha_k$ can not be 1-random.

The main theorem of this paper is Theorem 4.13. Notice that the proof is much simpler and more direct than a version of universality. This means that optimality is a more natural notion.

The characterization of $\mu(V_n)$ for an optimal ML-test $\{V_n\}$ implied another difference between universality and optimality. A difference between universality and optimality is known for martingales. We need to study further relation between tests and martingales.

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References

[1] C. Calude, P. Hertling, B. Khoussainov, and Y. Wang. Recursively enumerable reals and Chaitin Omega numbers. *Theoretical Computer Science*, 255(1):125–149, 2001.

[2] G. Davie. The Borel-Cantelli lemmas, probability laws and Kolmogorov complexity. *Annals of Probability*, 29(4):1426–1434, 2001.

[3] R. Downey, E. Griffiths, and G. LaForte. On Schnorr and computable randomness, martingales, and machines. *Mathematical Logic Quarterly*, 50(6):613–627, 2004.

[4] R. Downey and D. R. Hirschfeldt. *Algorithmic Randomness and Complexity*. Springer, Berlin, 2010.

[5] A. Kolmogorov. *Grundbegriffe der Wahrscheinlichkeitsrechnung*. Springer, 1933.

[6] A. N. Kolmogorov. Sur la loi forte des grands nombres. *CRAS Paris*, 191:910–912, 1930.

[7] A. Kučera and T. Slaman. Randomness and recursive enumerability. *SIAM Journal on Computing*, 31(1):199–211, 2002.

[8] L. A. Levin. Laws of information conservation (nongrowth) and aspects of the foundation of probability theory. *Problems of Information Transmission*, 10:206–210, 1974.

[9] L. A. Levin. Randomness consevation inequalities: Information and independence in mathematical theories. *Information and Control*, 61(1):15–37, 1984.

[10] M. Li and P. Vitányi. *An introduction to Kolmogorov complexity and its applications*. Graduate Texts in Computer Science. Springer-Verlag, New York, third edition edition, 2009.

[11] P. Martin-Löf. The Definition of Random Sequences. *Information and Control*, 9(6):602–619, 1966.

[12] W. Merkle, N. Mihailovic, and T. Slaman. Some results on effective randomness. *Theory of Computing Systems*, 39(5):707–721, 2006.

[13] A. Nies. *Computability and Randomness*. Oxford University Press, USA, 2009.

[14] C. Schnorr. A unified approach to the definition of a random sequence. *Mathematical Systems Theory*, 5:246–258, 1971.

[15] R. Solovay. Draft of paper (or series of papers) on Chaitin's work. unpublished notes, May 1975. 215 pages.

[16] V. G. Vovk. The law of the iterated logarithm for random Kolmogorov, or chaotic, sequences. *Theory Probab. Appl.*, 32:413–425, 1987.